

# Power Series Solutions of Differential Equations: Programming and Experimenting with Technology

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***Abstract.** We build a procedure of progressive programming using a symbolic and graphic calculator that does not abandon in an abrupt way traditional pencil and paper techniques to solve differential equations with power series. Basic idea is the construction of one or several small pieces of programs during the common tasks solutions in this topic of a differential equations course for an engineering career student. These programs allow us avoid complicated manual calculations and to achieve epistemological and didactical objectives in several stages during the teaching-learning process.*

## 1. Introduction.

One of current approaches in mathematical research is the so called experimental mathematics [1]. This approach recognizes the value of experimental exploration of conjectures, adoption of informal procedures and careful study of experimental results to test conjectures and to attempt rigorous proofs whenever it is possible and interesting.

Following the experimental trend in teaching mathematics, we encourage students to use technology in order to help them to formulate and to verify conjectures, to discover patterns, to explore, to make a synthesis of computational results and to appreciate proof and experiments as functional tools.

Implementation and development of experimental methods drastically change the way mathematics is taught and learned, imposing new demands to methodology, evaluation and curricula that should finally develop in students new abilities related with inductive and algorithmic thought, pattern's discovery, critical analysis of experimental results, construction and validation of conjectures and experimental search for formal proof. In this paper we intend to show some examples of the use of technology and the introduction of programming activities in order to develop such kind of mathematical abilities in a regular course of differential equations for an engineering career.

The transposition [2] of experimental approach with technology to mathematical education is not an easy process as J. B. Lagrange outlines in [3]. Lagrange argues that although it is possible to transpose the numerical and experimental part of experimental mathematics to teaching-learning process with certain success, the transposition of algorithmic thought and programming process present a bigger number of difficulties. Introducing a real algorithmic dimension by using calculators in mathematics teaching- leaning process implies a great change. "...a real teaching of algorithms is not feasible today because the traditional cultural kernel underpinning curricula is resistant, and too few research studies and experiments haven been undertaken." [3]

The basic motivation for our paper resides in the didactic experience that was carried out during two years at the Institute of Basic Sciences of the Faculty of Engineering of the Universidad Diego Portales, in which the use of a sophisticated calculator as Classpad300 Plus was used as an experimental tool in a formal plan of implementing technology in mathematics in the cycle of basic mathematic courses. This calculator provided us with a set of useful applications and with a basic programming language that use an easy and intuitive syntax [4]. The general reasons for having used mainly this calculator instead of strong mathematical computational software can be summarized in the following points:

- It is a very portable device that does not request special computation laboratory rooms.
- Prices are usually much lower than a computer.
- Using keys instead of commands optimize the use of time, because it is not necessary to memorize and to type special syntax in the computer.
- Calculators use friendly logic that involves a quick adaptation of students to the operational features of the calculator and his way of working

Although, today many of those points seem to be already solved, due to the existence of cheap, powerful and friendly laptops, we had an additional and main philosophical basis for having decided to use calculators:

*The importance in implementing technology in mathematic education should not be founded just in the use of strong software. The main reason should be based on the idea of being able to optimize the implementation of technology in teaching learning process education through a mutual powering between mathematics and technology. Whenever mathematics need to be developed by powering an adequate computational tool, skills based in programming should be developed at the earlier educational period and, on the other side, whenever technology need to be developed and improved by the creation of special mathematics, new scientific results and adequate theorems has to be created. The main reason should be the preparation for a future with people able to incorporate technology - not as an external device independent of academic or intellectual work- but as an integral part of mathematical thinking [5].*

Examples of applying this philosophy basis in using calculators can be found in our previous work in [6], [7], [8], [9] and [10].

Regarding programming activities, in spite of real difficulties, there are several reasons to introduce them as regular activities in the cycle of university mathematics that we consider important. Some of them are:

- Construction of algorithms and programs impose a correct definition of objects and concepts used in mathematics to assure an intelligent dialogue with the computational tool. Student, for example, should clearly understand concepts such as: list, vector, matrix, expression, function, equation, etc.
- Mathematical activity becomes creative and motivational since student realizes that they can create new functions (user's functions), commands, programs, subroutines, etc. in mathematical problem – solving situations.
- The construction of programs discovers new learning situations which can be used by the teacher in benefit of an effective communication of mathematical knowledge.
- Programs don't constitute black boxes but rather become the field of action for mathematics.

- New requirements appear such as effectiveness, algorithmic thought, economy and precision in concepts, accuracy in the syntax, complexity, etc.

Our experience in teaching at Universidad Diego Portales has included the use of calculators in the very first courses of Algebra and of Calculus of one variable, developing in this way the basic necessary skills, such as, working with different windows of the calculator and its interrelations, managing special syntax, adequating and using variables and parameters [11], paying special attention in the configuration of the view-window and the use of an adequate set up for the calculator according to mathematical necessities when it comes to analysis and graphifications.

The work in programming with calculators were first established when it was necessary to introduce or define user's functions in Calculus as new special commands of the calculator, as well as in Algebra when the inclusion of automatic commands were constructed for mechanical or repetitive calculations such as determination of central terms in a Newton's binomial, roots of complex numbers using De Moivre's formula, formulas for arithmetic and geometric progressions, etc. Students of Calculus of Several Variables and Differential Equations that had already achieved advanced skills in using calculators as tools in the resolution of tasks, were supposed to be able to use more complex syntax and to construct programmes with bigger challenges, and for that reason the requirements for them became also higher and more demanding.

This paper is intended to motivate teachers in introducing programming and algorithmic thought as natural and necessary skills beside traditional ones, whether following a regular sequence of contents or a most modern approach. We will illustrate the use of programming with Classpad300 calculator, giving some examples of its application in power series solutions of differential equations.

Techniques to solve differential equations with power series [12] are tedious, laborious and often contribute little to cognitive and didactic aspect in this topic of differential equations. Lack of time and density of the curricula of this subject doesn't allow the study of many cases and to really understand the method by itself. Traditional techniques lack pragmatic value and possess a doubtful epistemic value. Usually with those techniques the students learn certain tricks but they often lose the view central idea.

To solve these difficulties we build a procedure of progressive programming that does not abandon in an abrupt way the traditional pencil and paper techniques, but rather at the beginning offer a complement of practical value to those techniques and progressively become into a tool to generate new knowledge.

Basic idea is the construction of one or several small pieces of programs with practical value which allow quick resolution of the task, diminishing annoying manual calculations, and on the other hand achieving the epistemological and didactical value in several stages, although all the stages could not necessarily be present in a simultaneous way. The very first of these stages is the creation of algorithms and the program, the second is the interpretation and critical analysis of the obtained results and the third is a comparison with other methods, for example by means of graphic visualizations.

In this article we have tried to follow traditional sequence in the contents of our paper to the one followed in usual courses of differential equations. In that sense, section 2 is dedicated to power series solutions of homogeneous differential equations in the neighborhood of an ordinary point. In section 3 we study non homogeneous equations, also in the neighborhood of an ordinary point. In section 4 we use the Frobenius Method to solve differential equations in the neighborhood of a singular regular point. Lastly in section 5 we study the solution of Legendre's differential equation. This solution is related with the so called Legendre's polynomials.

In each one of these sections, programs for Classpad300 are built to find and to draw the solutions, trying to be the clearest and intuitive possible, so that the students can be concentrated most of the time on the mathematical problem of interest.

In spite of using mainly calculator in the tasks, we add as an appendix a list of codes and programs for the very same examples shown with Classpad300 to be performed in Maple as well.

## 2. Power series solutions of differential equations in the neighborhood of an ordinary point.

### Task 2.1

It's asked to students, find power series solution of differential equations, using traditional way with pencil and paper. Once the students discover the recurrence formula for the series coefficients, it's requested to build a Classpad300 program to automatically generate terms of the series. The question is the following one:

Use a power series in the neighborhood of ordinary point  $x = 0$  to demonstrate that general solution of the equation  $y'' + xy' + y = 0$ , can be written by means of the following power series:

$$y = A_0 \left( 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \dots \right) + A_1 \left( x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \dots \right)$$

Firstly find the recurrence relation for coefficients and construct with it a program using the calculator to automatically generate coefficients of the series.

### **Solution**

We look for the solution in form of power series:

$$y = \sum_{i=0}^{\infty} A_i x^i, \quad y' = \sum_{i=0}^{\infty} i A_i x^{i-1}, \quad y'' = \sum_{i=0}^{\infty} i(i-1) A_i x^{i-2},$$
 we replace these sums in the equation

$$\sum_{i=2}^{\infty} i(i-1) A_i x^{i-2} + \sum_{i=0}^{\infty} i A_i x^i + \sum_{i=0}^{\infty} A_i x^i = 0$$
 and make the following change of variable

$i - 2 = n \Rightarrow i = n + 2$  in the first of the sums:

$$\sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} x^n + \sum_{n=0}^{\infty} n A_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

Now gather the coefficients with equal exponents

$$\sum_{n=0}^{\infty} [(n+2)(n+1) A_{n+2} + (n+1) A_n] = 0$$

From where we obtain the following recurrence formula:

$$(n+2)(n+1)A_{n+2} + (n+1)A_n = 0 \Rightarrow A_{n+2} = -\frac{A_n}{n+2}, \quad n \geq 0.$$

The figure 1 shows a program that allows the generation of the coefficients of the series, once it is known the recurrence relation. The program contains the parameter "N" which indicates the number of times that recurrence formula is applied.

The first two coefficients  $A_0$  and  $A_1$  are arbitrary and the program includes them by default. Therefore, the number of coefficients obtained when we apply the program is  $N + 2$ .

The figures 2 and 3 show the use of the program for  $N = 4$ .

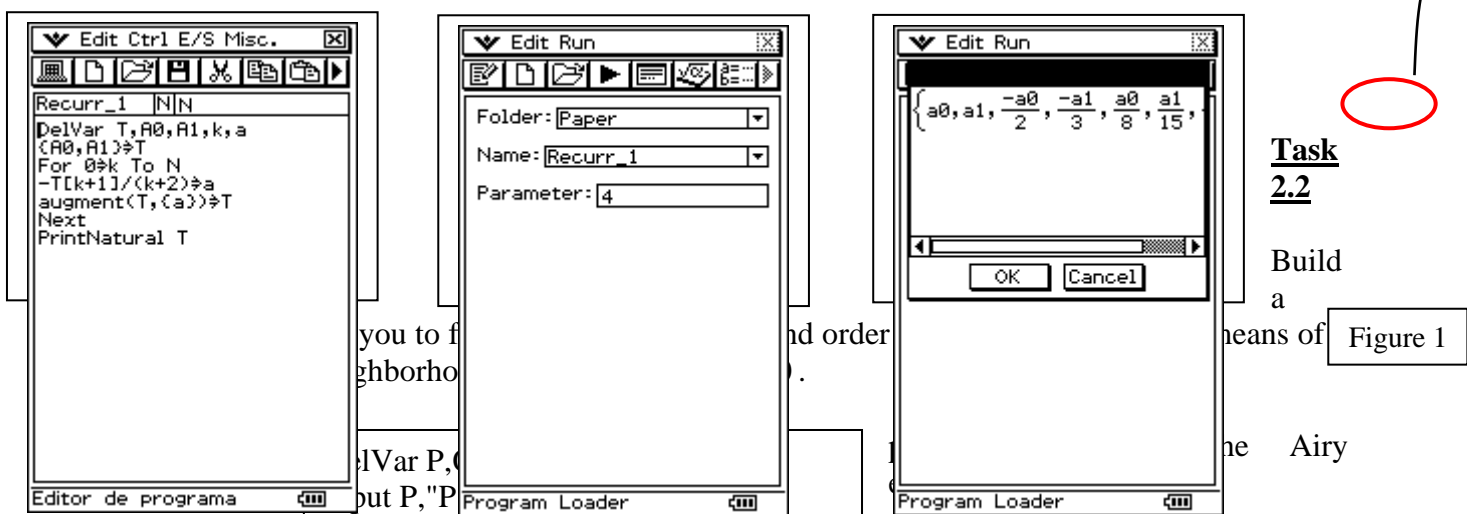
The series take the form:

$$y = A_0 + A_1x - \frac{1}{2}A_0x^2 - \frac{1}{3}A_1x^3 + \frac{1}{8}A_0x^4 + \frac{1}{15}A_1x^5 + \dots$$

Or in another way:

$$y = A_0 \left( 1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots \right) + A_1 \left( x - \frac{x^3}{3} + \frac{x^5}{15} + \dots \right)$$

Parameter



**Task 2.2**

Build a

Figure 1

ne Airy

**Solution**

The box below allows finding

```

Input Q, "Q"
Input N, "N>3"
((P|x=0) × a1 + (Q|x=0) × a0) / 2 ⇒ A
{ a0, a1, A } ⇒ T
(P × diff(h(x), x) + Q × h(x)) ⇒ E
For 1 ⇒ k To N-3
simplify(diff(E, x)) ⇒ E
simplify(E) | diff(h(x), x) = a1 ⇒ E1
For 2 ⇒ j To k+1
simplify(E1) | diff(h(x), x, j) = j! × T[j+1] ⇒
E1
Next
simplify(E1) | h(x) = a0 | x=0 ⇒ E2
E2 / ((k+2)!) ⇒ f
augment(T, {f}) ⇒ T
Next
PrintNatural T
    
```

shows the program "SerieP" which the coefficients of partial

$$\text{series } y = \sum_{n=0}^N a_n x^n, \text{ with } N > 3.$$

To use this program, the differential equation should be written in the form:

$$y'' = P(x)y' + Q(x)y \quad (2.1)$$

The user should introduce the functions  $P(x)$  and  $Q(x)$ , besides the number  $N$  of terms to consider in the partial series.

The program works in the following way. Suppose that there is a solution of a differential equation (2.1) given by  $y = h(x)$ . Assuming that function  $h(x)$  has a Taylor series in the neighbourhood of an ordinary point  $x_0 = 0$  which converges for  $|x| < \rho$ , with  $\rho > 0$ , i.e.  $h(x)$  can be written in the

form  $h(x) = \sum_{n=0}^{\infty} a_n x^n$ , where the coefficients can be calculated by the relationship  $a_n = \frac{h^{(n)}(0)}{n!}$ .

Here  $h^{(n)}(0)$  mean that the function is differentiated  $n$  times and then set  $x$  equal to 0.

To compute these coefficients we must be able to find  $h^{(n)}(0)$  from the differential equation (2.1). If we are only interested in general solution of the differential equation, then the coefficients  $a_0 = h(0)$  and  $a_1 = h'(0)$  are arbitrary. The equation (2.1) only specifies coefficients  $a_n$  for  $n \geq 2$ . Since  $h(x)$  is a solution of equation (2.1) we have:

$$h''(x) = P(x)h'(x) + Q(x)h(x) \quad (2.2)$$

Using (2.2) we find that:

$$a_2 = \frac{h''(0)}{2!} = \frac{P(0)h'(0) + Q(0)h(0)}{2} = \frac{P(0)a_1 + Q(0)a_0}{2} \quad (2.3)$$

In the program the relationship (2.3) is written with the following syntax:

$((P|x=0) \times a1 + (Q|x=0) \times a0) / 2 \Rightarrow A$

The results of coefficient's calculations are stored in the list "T". So far we have three coefficients in the list:

$$\{a_0, a_1, A\} \Rightarrow T$$

To compute  $a_3$  we should differentiate (2.2) one more time and set  $x = 0$ . The process repeats for the other coefficients in a similar way. In the program, the right side of the relationship (2.2) is stored in the variable "E":

$$(P \times \text{diff}(h(x), x) + Q \times h(x)) \Rightarrow E$$

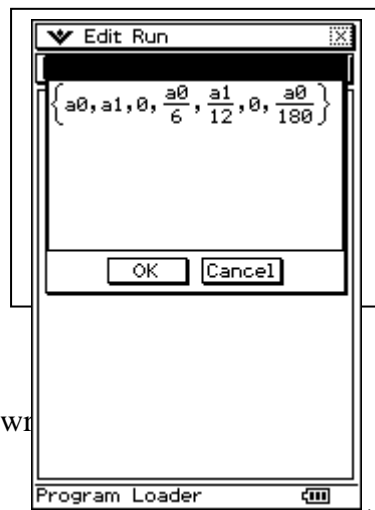
The variable "E" is updated each time we differentiate.

To obtain the coefficients in an automatic way a cycle is being built that considers the number "N" of terms requested by the user. As we already know three coefficients in the list "T" before the cycle begins to work, we should take  $k$  from  $k = 1$  to  $k = N - 3$ .

The lower order derivatives of the function  $h(x)$  that appear every time the expression "E" is differentiated are replaced with the already obtained coefficients from the previous cycles. Every new obtained coefficient "f" is added to the list "T" with a command:

$$\text{augment}(T, \{f\}) \Rightarrow T$$

- a. To solve Airy equation using program "**SeriesP**", we write  $y'' = xy$ . This implies that  $P(x) = 0$  and  $Q(x) = x$ . The result for  $N = 7$  is shown in figure 4.



This result can be written as

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \frac{1}{180} a_0 x^6 + \dots$$

Or in another way:

$$y = a_0 \left( 1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \dots \right) + a_1 \left( x + \frac{1}{12} x^4 + \dots \right)$$

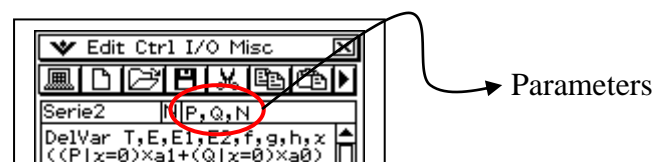
### Task 2.3

Build a program that allows you to obtain and draw the graph of particular solutions of differential equation  $y'' = P(x)y' + Q(x)y$ , using power series in the neighborhood of ordinary point  $x=0$  and given specific values for coefficients  $a_0$  and  $a_1$ .

- To study graphically the convergence of the obtained series, consider the differential equation  $y'' + y = 0$ , with initial conditions  $y(0)=0, y'(0)=1$ . Find solution for this differential equation using the program made by you and draw the graph of this solution with growing values of  $N$ .
- Explain the obtained results.

### **Solution**

The program "**SerieGra**" is shown in the box bellow.



```

DelVar P,Q,a,b,T,T1,sp,B,Z,D,U
Input P, "P"
Input Q, "Q"
Input N, "N>3"
Input a, "a0"
Input b, "a1"
Serie2(P,Q,N)
T|a0=a|a1=b => T1
a => sp
For 2 => k To N
sp+T1[k] * x^(k-1) => sp
Next
ClrGraph
ClearSheet 1
SetSimulGraph On
DefaultSetup
ClrGraph
Input B, "Xmin", "ViewWindow"
Input Z, "Xmax", "ViewWindow"
Input D, "Ymin", "ViewWindow"
Input U, "Ymax", "ViewWindow"
Wait 1
ViewWindow B,Z,1,D,U,1
GTSelOn 1
GraphType "y="
Print "Partial Sum:"
Print sp
Define y1(x)=sp
DrawGraph
DelVar P,Q,a,b,T,T1,B,Z,D,U
    
```



The program “**SerieGra**” uses as a subroutine another program named “**Serie2**”, which is shown in the figure 5. Notice that this last program uses the parameters “P, Q, N” and return a list “T” with N coefficients of the partial sum, that in turn will be used by the program “**SerieGra**” to build a polynomial and to draw it.

To use the program, the differential equation should be written in the form:

$$y'' = P(x)y' + Q(x)y.$$

The initial conditions determine the first two coefficients. We have  $y(0) = a_0$  and  $y'(0) = a_1$ .

The program “**SerieGra**” makes a call to the subroutine “**Serie2**” using the command:

Serie2(P,Q,N)

This subroutine gives a list of coefficients. The coefficients  $a_0$  and  $a_1$  are specified by the given initial conditions. With this list the partial series is built and later is stored in the variable “sp”. The partial series can be drawn using the commands:

Define y1(x) = sp  
DrawGraph

But before, the user should define the appropriate window to show the graph. This window can be modified later on in the menu “Graph&Table” of the calculator.

We know by other methods that general solution of the equation  $y'' + y = 0$  is given by:

$$y = A \cos x + B \sin x.$$

To find the particular solution which satisfies given initial conditions we should write

$$y(0) = A \cos 0 + B \sin 0 = A = 0$$

$$y'(0) = B \cos 0 = B = 1.$$

Therefore the particular solution is  $y = \sin x$ .

Student should know at this point that in general it is not always possible to find solutions of differential equations with the help of elementary functions; it is in these cases when power series method is truly useful.

We will use equation  $y'' + y = 0$  with initial conditions  $y(0) = 0$ ,  $y'(0) = 1$ , for which particular solution we know, to study the convergence of the obtained by the program “**SerieGra**” series.

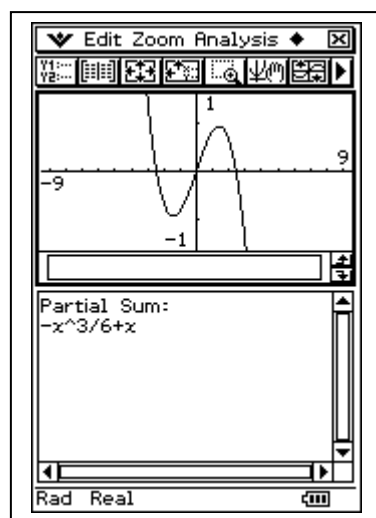


Figure 6

The following data should be entered to the program “**SerieGra**”:

- “P”  $\Rightarrow$  0
- “Q”  $\Rightarrow$  -1
- “N>3”  $\Rightarrow$  5
- “a0”  $\Rightarrow$  0

- “a1”  $\Rightarrow$  1
- “Xmin”  $\Rightarrow$  -9.5,
- “Xmax”  $\Rightarrow$  9.5,
- “Ymin”  $\Rightarrow$  -1.6,
- “Ymax”  $\Rightarrow$  1.6

to obtain the result showed in figure 6.

Repeating this procedure for the values  $N = 7$  y  $N = 10$  can be observed a convergence of partial series to the function  $y = \sin x$ , which is the particular solution to differential equation  $y'' + y = 0$ , with initial conditions  $y(0) = 0$ ,  $y'(0) = 1$ . (See figures 7-9)

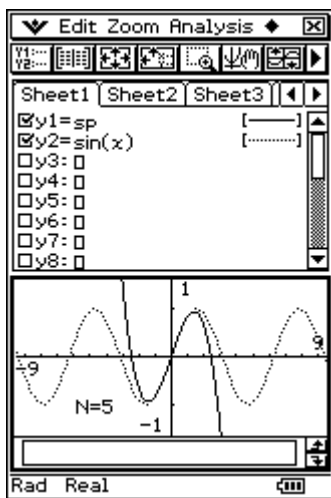


Figure 7

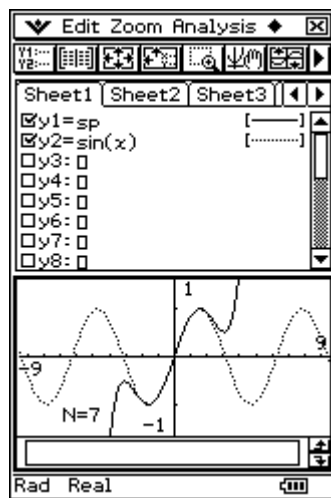


Figure 8

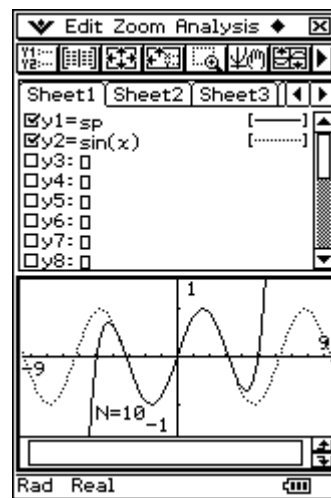
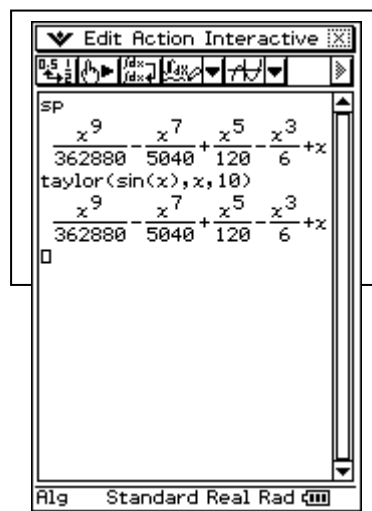


Figure 9

With these results, together with the graphs, the student is able to understand in a better way the following situation. When we apply the method of power series in a neighborhood of an ordinary point to find solution of differential equation, what we really do is finding the Taylor series of the solution in the neighborhood of this point. The figure 10, shows the partial result for  $N = 10$  and the corresponding Taylor series for the function  $y = \sin x$  in the neighborhood of  $x = 0$ . Notice that the result of applying the program, is kept in the variable “sp” that we can use in future calculations or graphics.



### 3. Non homogeneous equations.

#### Task 3.1

Modify the program “**SerieP**” to consider non homogeneous equations, i.e. equations in the form  $y'' + P(x)y' + Q(x)y = F(x)$ .

- a. With this new program, find first 7 terms of the solution with power series of the equation  $y'' - 2x^2y' + 4xy = x^2 + 2x + 2$ .

#### Solution

The modified program, which we have denominated “**SerNonHom**”, is shown in the box below. The modifications that were done are marked by small green rectangles.

To apply this program we should write the equation in the form:

$$y'' = P(x)y' + Q(x)y + F(x)$$

The modifications done constitute a simple generalization of the relationships (2.2) and (2.3) that consider an additional function  $F(x)$ .

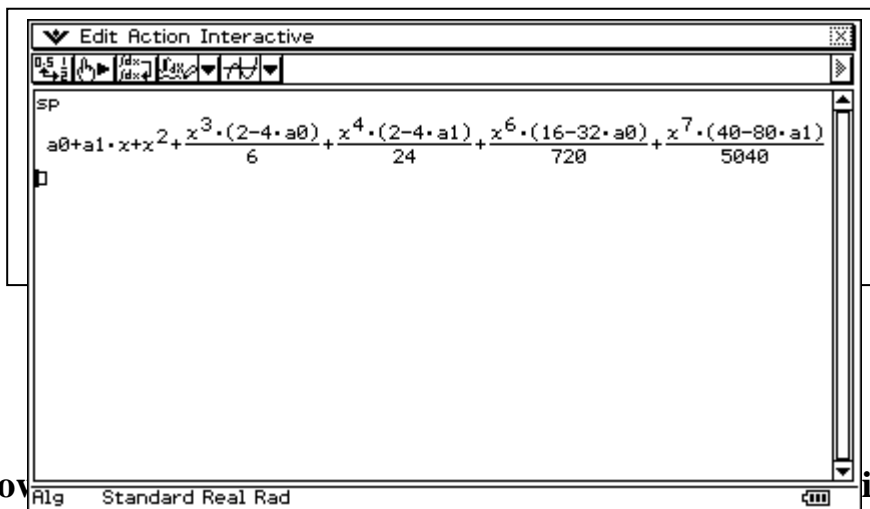
For example, for the equation  $y'' - 2x^2y' + 4xy = x^2 + 2x + 2$  we should write:

$$y'' = \overset{P}{2x^2} y' - \overset{Q}{4x} y + \overset{F}{x^2 + 2x + 2}$$

The result of using the program “**SerNonHom**” for this equation with  $N = 7$  is shown in figure 11.

```

DelVar P, Q, N, T, E, E1, E2, x, sp,A,B
Input P,"P"
Input Q,"Q"
Input F, "F"
Input N, "N>3"
((P|x=0)×a1+(Q|x=0)×a0+F|x=0)/2 ⇒ A
{a0,a1,A } ⇒ T
(P×diff(h(x),x)+Q×h(x)+F) ⇒ E
For 1 ⇒ k To N-3
simplify(diff(E,x)) ⇒ E
simplify(E)|diff(h(x),x)=a ⇒ E1
For 2 ⇒ j To k+1
simplify(E1)|diff(h(x),x,j)=j!×T[j+1] ⇒ E1
Next
simplify(E1)|h(x)=a0|x=0 ⇒ E2
E2/((k+2)!) ⇒ B
augment(T,{ B }) ⇒ T
Next
a0 ⇒ sp
For 2 ⇒ k To N
sp+T[k]×x^(k-1) ⇒ sp
Next
PrintNatural sp
DelVar P,Q,N,E,E1,E2
    
```



**4. Power Series**

**Task 4.1**

Build a program to find the indicial equation and to solve it, in the case of solutions with power series in the neighborhood of a singular regular point of differential equations in the form:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- a. Use the program to calculate the roots of the indicial equation for the differential equation  $2x^2y'' - xy' + (1+x)y = 0$

**Solution**

We will suppose without loss of generality that  $x=0$  is a singular regular point for the equation  $P(x)y'' + Q(x)y' + R(x)y = 0$ . We divide the equation by  $P(x)$  and multiply by  $x^2$  and obtain:

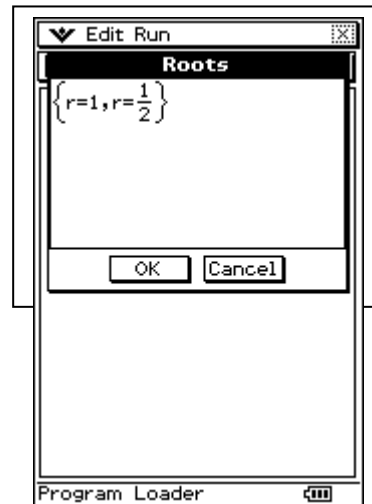
$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)} \text{ y } q(x) = \frac{R(x)}{P(x)}$$

The indicial equation is determined by means of the equation  $F(r) = r(r-1) + p_0r + q_0 = 0$ , where  $p_0 = \lim_{x \rightarrow 0} x \cdot p(x)$  and  $q_0 = \lim_{x \rightarrow 0} x^2q(x)$ .

The program “**Indicial**” which uses these formulas is shown below in the box. The roots of the indicial equation for the equation  $2x^2y'' - xy' + (1+x)y = 0$  are shown in the figure 12.

```

DelVar P,Q,R,p0,r,q0,S
Input P, "P"
Input Q, "Q"
Input R, "R"
lim(x × Q/P,x,0) ⇒ p0
lim(x^2 × R/P,x,0) ⇒ q0
solve(r × (r-1)+p0 × r+q0=0,r) ⇒ S
PrintNatural S, "Roots"
DelVar P,Q,R,p0,r,q0
    
```



**Task 4.2**

Build a program to find “N” coefficients of the power series in the neighborhood of a singular regular point of a differential equation of the form  $P(x)y'' + Q(x)y' + R(x)y = 0$ , if the roots  $r_1$  and  $r_2$  for the indicial equation are known. (*Frobenius Method*)

- a. Use the program to find “N = 5” terms of the general solution of the equation:

$$2x^2y'' - xy' + (1 + x^2)y = 0$$

**Solution**

In the box below the program “**Frobe**” is shown. This program allows calculating the solution of a differential equation  $P(x)y'' + Q(x)y' + R(x)y = 0$  with a power series in the neighborhood of a singular regular point, using the so called *Frobenius Method*.

If the equation  $P(x)y'' + Q(x)y' + R(x)y = 0$  posses a singular regular point  $x_0 = 0$  then it can be written in the following form:

$$x^2y'' + x \left[ x \frac{Q(x)}{P(x)} \right] y' + \left[ x^2 \frac{R(x)}{P(x)} \right] y = 0 \tag{4.1}$$

where the expressions inside the brackets have Taylor series in the neighborhood of the singular regular point  $x_0 = 0$ . In the program “**Frobe**”, Taylor series for these functions are constructed using the following commands:

```

taylor(x × Q/P,x,N) ⇒ p
taylor(x^2 × R/P,x,N) ⇒ q
    
```

To continue with the calculations we will need the coefficients of these series. To do this program uses the subroutine “**CoefPx**” (see figure 13), to which we should introduce two parameters: a polynomial and its degree, and returns a list (contained in the variable “T”) with the coefficients of the polynomial. The coefficients for these functions are stored in the lists “Tp” and “Tq”.

This way the equation (4.1) takes the form:

$$x^2 y'' + x \left[ \sum_{k=0}^{\infty} p_k x^k \right] y' + \left[ \sum_{k=0}^{\infty} q_k x^k \right] y = 0 \quad (4.2)$$

where the coefficients  $p_k \equiv \text{Tp}[k+1]$  and  $q_k \equiv \text{Tq}[k+1]$  are already calculated.

In the *Frobenius Method* we look for the solution in the form  $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ .

By substituting the expressions for  $y$ ,  $y'$  and  $y''$  in equation (4.2), multiplying the infinite series together and then collecting terms, we obtain: [3]

$$a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0 \quad (4.3)$$

where  $F(r) = r(r-1) + p_0 r + q_0$ . According to (4.3) we have  $F(r) = 0$ , which is the indicial equation already treated in the previous task. We also obtain that coefficients  $a_n$  can be calculated by means of the formula:

$$a_n = - \frac{\sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}]}{F(r+n)} \quad (4.4)$$

The numerator in (4.4) is calculated by the program and stored in the variable “su”. The denominator is defined by means of the expression:

$$\boxed{x \times (x-1) + \text{Tp}[1] \times x + \text{Tq}[1] \Rightarrow \text{FI}}$$

Each new coefficient is temporarily stored in the variable “g”:

$$\boxed{-\text{su}/(\text{FI}|_{x=(\text{ra}1+n)}) \Rightarrow \text{g}}$$

where “ra1” is one of the roots of indicial equation. Then “g” is added to the list “T”, using the command:

$$\boxed{\text{augment}(\text{T}, \{\text{g}\}) \Rightarrow \text{T}}$$

Finally with this list the partial series is built.

Notice that the program should be applied twice, once for each root of the indicial equation.

```

DelVar P,Q,R,N,p,q,ra1,T
Input P, "P"
Input Q, "Q"
Input R, "R"
Input N, "Number of terms"
Input ra1, "Root Indicial Eq."
taylor(x×Q/P,x,N)⇒p
taylor(x^2×R/P,x,N)⇒q
CoefPx(p,N)
T⇒Tp
DelVar T
CoefPx(q,N)
T⇒Tq
DelVar T
x×(x-1)+Tp[1]×x+Tq[1]⇒FI
{a0}⇒T
For 1⇒n To N
0⇒su
For 0⇒k To n-1
su+T[k+1]×((ra1+k)×Tp[n-k+1]+Tq[n-
k+1])⇒su
Next
-su/(FI|x=(ra1+n)) ⇒g
augment(T,{g})⇒T
Next
a0⇒res
For 2⇒j To N
res+T[j]×x^(j-1)⇒res
Next
PrintNatural x^(ra1)×res,"Power Series"
    
```

a. To solve the equation  $2x^2y'' - xy' + (1+x^2)y = 0$  using the *Frobenius method*, we take  $P = 2x^2$ ,  $Q = -x$ ,  $R = 1+x^2$  and find the roots of the indicial equation using the program “**Indicial**”. For this equation we obtain the roots  $r_1 = 1$  and  $r_2 = \frac{1}{2}$ .

Next we use the program “**Frobe**” with the first of the roots,  $r_1 = 1$ . The result is shown in the figure 14.

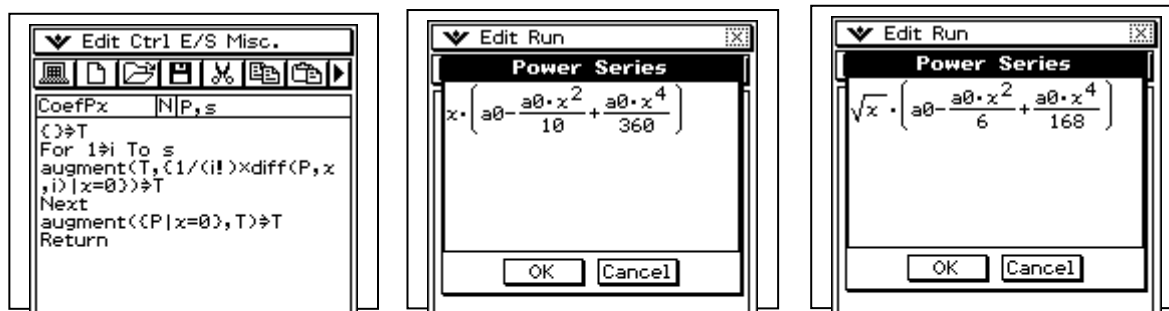




Figure 13

Figure 14

Figure 15

For the second root  $r_2 = \frac{1}{2}$  the program yields the result shown in the figure 15.

Putting together both results and taking as usual  $a_0 = 1$  we write the general solution of the equation in the following form:

$$y = Ax \left( 1 + \frac{x^2}{10} + \frac{x^4}{360} + \dots \right) + B\sqrt{x} \left( 1 - \frac{x^2}{6} + \frac{x^4}{168} + \dots \right)$$

## 5. The Legendre's polynomials.

In this section we show with several examples how we can study some of the basic properties and definitions of the Legendre's polynomials using a calculator. We know that Legendre's polynomials  $P_n(x)$  are solutions of the differential equation which is known as the Legendre's

$$\text{Equation } (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

On the other hand, these polynomials constitute an orthogonal infinite base and can be used to express other functions in form of series. Our first activity with the students is to define a user's function that allows to generate these polynomials.

### Task 5.1

Construct a user's function to calculate the Legendre's polynomials of any order  $n$ .

### **Solution**

To calculate the Legendre's polynomials of order  $n$ , we use the "Rodrigues's formula". We define the following user's function:

$$\text{Define library}\backslash\text{Legendre}(n)=\text{expand}\left(\frac{1}{2^n n!} \times \text{diff}\left(\left(x^2-1\right)^n, x, n\right)\right)$$

The figures 16 - 18 shows how to generate the polynomials with help of this formula.

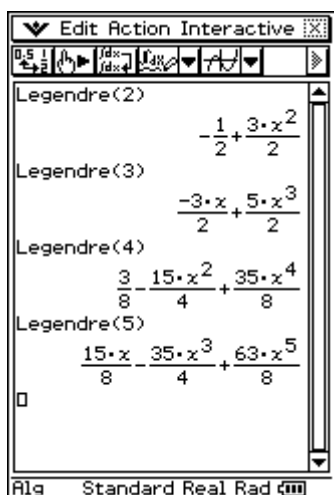


Figure 16

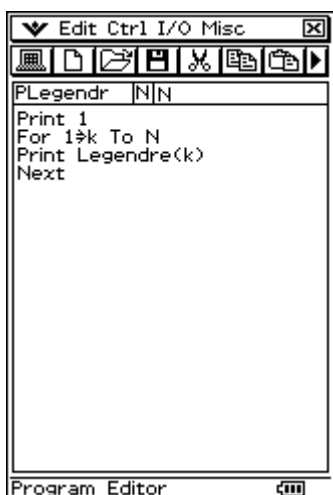


Figure 17

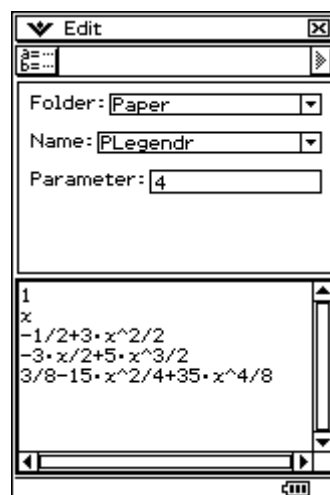


Figure 18

**Task 5.2**

Let us consider the vector space of polynomials  $P_{[-1,1]}(x)$  defined in the interval  $x \in [-1,1]$  with the inner product  $(f, g) = \int_{-1}^1 f(x) \cdot g(x) dx$ . Investigate with the help of a calculator the property of orthogonality of the Legendre's polynomials.

- a. Formulate a conjecture on the result of the operation  $(P_n, P_n) = \|P_n\|^2$ , where  $P_n(x)$  is a Legendre's polynomial of order  $n$ .

**Solution**

Firstly we build user's function that allows to calculate the inner product in the vector space  $P_{[-1,1]}(x)$

Define library\Pint(a,b)=  $\int_{-1}^1 (a \times b, x, -1, 1)$

The figure 19 shows some calculations that suggest the orthogonality of the Legendre's polynomials

- a. The figure 20 shows the calculations for  $(P_n, P_n) = \|P_n\|^2$ .



Figure 19

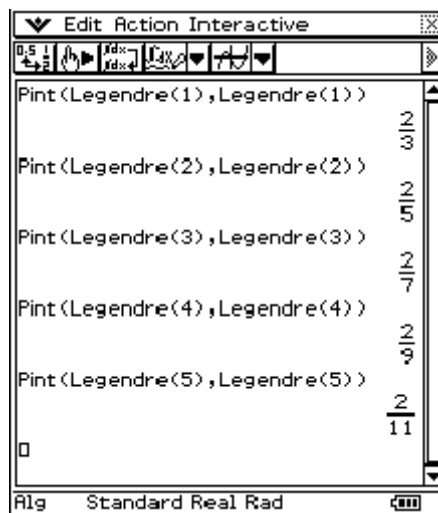


Figure 20

The student observe that the results indicate that a fraction is always obtained with 2 in the numerator and an odd number in the denominator. It is easy to notice that the odd numbers that appear in the denominator are obtained taking  $2n+1$ , where  $n$  is the polynomial order. The calculations suggest that  $\|P_n\|^2 = \frac{2}{2n+1}$

### Task 5.2

Consider the vector subspace of the polynomials with degree less than 4 in the interval  $x \in [-1,1]$ . The Legendre's polynomials until order four form a base of this subspace. Express the polynomial  $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$  as a linear combination of the Legendre's polynomials.

### Solution

The polynomial  $f(x)$  is expressed as a linear combination of Legendre's polynomials until order four by means of the relationship  $f(x) = \sum_{n=1}^4 c_n P_n(x)$ . Bearing in mind that Legendre's polynomials are orthogonal, the coefficients  $c_n$  are calculated by means of the formula:

$$c_n = \frac{(f, P_n)}{\|P_n\|^2} = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x) dx,$$

for  $n = 1, 2, 3, 4$ .

The figures 22-23 show the calculations carried out with the help of the program "**CnLege**" given in the figure 21. The function is expressed then in the form:

$$f(x) = -\frac{32}{15} P_0(x) + \frac{1}{5} P_1(x) + \frac{40}{21} P_2(x) + \frac{4}{5} P_3(x) + \frac{8}{35} P_4(x)$$

```

Edit Ctrl E/S Misc.
CnLege |N|f,N
Print 1/2*Pint(f,1)
For 1=k To N
Print (2*k+1)/2*Pint(f,Lege
ndre(k))
Next
    
```

We can check that the result is:

$$\text{simplify}(-32/15 + 1/5 \times \text{Legendre}(2) + 4/5 \times \text{Legendre}(3) + 8/35 \times \text{Legendre}(4))$$

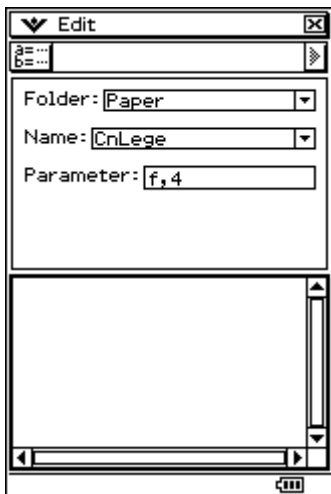


Figure 22

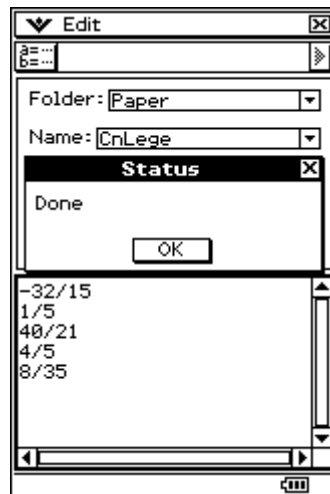


Figure 23

**Task 5.3**

Another way to build the Legendre's polynomials is using the generating function  $G(x,r) = \frac{1}{\sqrt{1-2xr+r^2}}$ , which can be written as a power series using the variable  $r$ , with the condition that  $r$  and  $x$  should be chosen to assure the convergence of the series. With the help of the calculator find the Taylor's series of the function  $G(x,r)$  and verify that the coefficients of  $r^n$  are just the Legendre's polynomials  $P_n(x)$ .

### Solution

The figure 24 shows the Taylor's series with two and three terms. Comparing with the figure 18 we see that the coefficients of  $r^n$  are exactly the Legendre's polynomials  $P_n(x)$ .

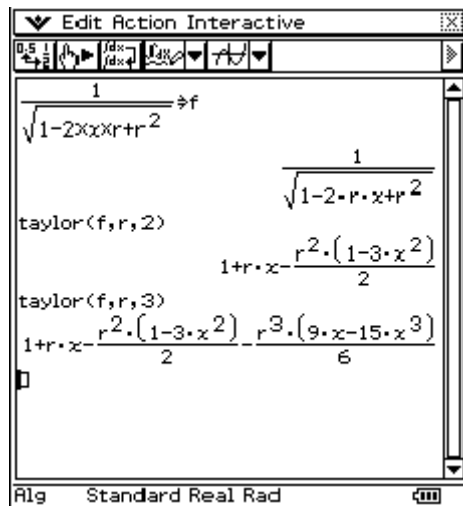


Figure 24

### Task 5.4

Construct a program that generates the Legendre's polynomials using the generating function  $G(x,r) = \frac{1}{\sqrt{1-2xr+r^2}}$ .

### Solution

The generating function  $G(x,r)$  can be written as a power series by mean of

$$G(r,x) = \sum_{n=0}^{\infty} P_n(x)r^n$$

where  $P_n(x)$  is the Legendre's Polynomial of order  $n$ .

To generate the Legendre's polynomials firstly we build a subroutine to calculate the coefficients of a polynomial. This subroutine which we denominate "**CoefP**" was already used in the task 4.2 and it is shown in the figure 13.

This subroutine uses two parameters: a polynomial with a variable  $r$  and its degree. It returns a list, extracting the coefficients of  $r^n$  in the polynomial.

The figure 25 shows the program "**GenLeg**" that allows to generates the Legendre's polynomials with degree smaller or equal than certain value N. An example of the program's application is shown in figures 26 and 27

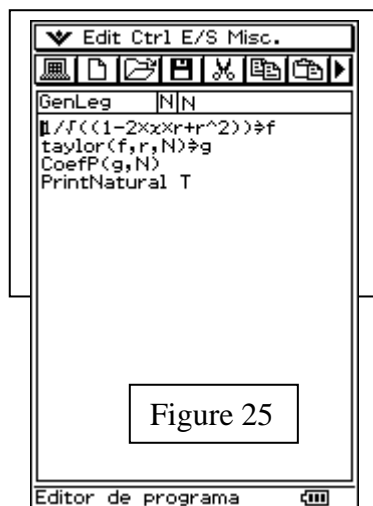


Figure 25

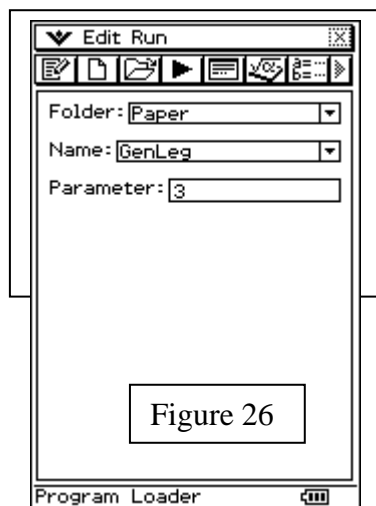
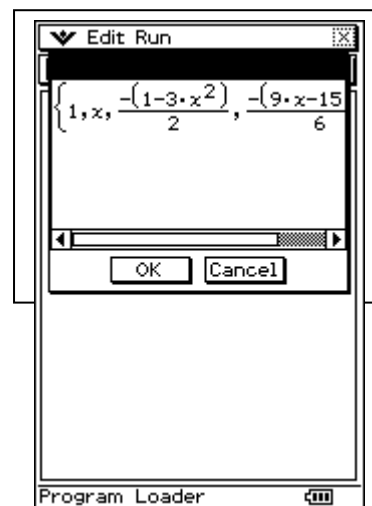


Figure 26



### Task 5.5

Using Gram – Schmidt procedure construct a program to orthogonalize sets of polynomials of the vector space  $P_{[-1,1]}(x)$  with the inner product defined by

$$(f, g) = \int_{-1}^1 f(x) \cdot g(x) dx .$$

If we denote by  $\{p_0(x), p_1(x), \dots, p_n(x), \dots\}$  the polynomials obtained by the orthogonalization of canonical polynomials  $\{1, x, \dots, x^n, \dots\}$ , show that Legendre's polynomials can be obtained using the formula:

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} p_n(x).$$

### Solution

The box below shows the program “**Ortoleg**” that allows to generate the Legendre's polynomials from the canonical polynomials

```

Input s, "Number of polynomials"
{} ⇒ T
For 1 ⇒ i To s
InputFunc p(x), "Polynomial"
augment(T, {p(x)}) ⇒ T
Next
For 1 ⇒ i To s
0 ⇒ h
For 1 ⇒ k To i-1
expand(h+Pint(T[i],T[k])/Pint(T[k],T[k])×T[k]) ⇒ h
Next
T[i]-h ⇒ T[i]
Next
{} ⇒ F
For 1 ⇒ k To s
k-1 ⇒ i
((2×i)!)/((2^i)×(i!)^2)×T[k] ⇒ F1

```

Additional questions about Legendre's polynomials using calculators can be done. We add a few of them in the following list in order to motivate the reader to work on it.

1. With the help of the calculator observe that  $P_{2n}(x)$  contains only even powers whereas  $P_{2n+1}(x)$  contains only odd powers of  $x$ . Why does this happen? Suggestion: Analyze in detail the Rodrigues's formula.

2. Legendre's polynomial satisfy the recurrence relation:

$$P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

for  $n \geq 1$  and all value of  $x$ . This result can even be valid for  $n=0$  if we consider  $P_{-1}(0) \equiv 0$ . Construct a program that uses the previous recurrence relation to generate the Legendre's polynomials.

3. With the help of the calculator observe that  $P_n(1) = 1, \forall n$ . Show this property using the recurrence relation given in the previous question.

4. With the help of the calculator observe that  $P_n(-x) = (-1)^n P_n(x)$ . Show this property.
5. Use the properties given in questions 3 and 4 to proof that  $P_n(-1) = (-1)^n$ .
6. Check with the help of the calculator that the Legendre's polynomials  $P_n(x)$  are solutions of the Legendre's equation.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

7. In electrodynamics the potential that produces an unitary charge in which the position is given by the vector  $\vec{r}'$  in a point with position vector  $\vec{r}$  is defined by the function  $\Phi = \frac{k}{|\vec{r} - \vec{r}'|}$ , where  $k$  is the electrical constant. Show that the function  $\Phi$  can be written in

the form of series by means of  $\Phi = k \sum_{n=0}^{\infty} \frac{r_{<}^n}{r_{>}^{n+1}} P_n(\cos \theta)$  where  $r_{<}$  ( $r_{>}$ ) is the smallest (greatest)

of the modules  $|\vec{r}'|$  or  $|\vec{r}|$ . Suggestion: Write  $\frac{k}{|\vec{r} - \vec{r}'|} = \frac{k}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}}$  and compare with the generating function given in the question 4 of proposed exercises.

8. The denominated *Associated Legendre's Polynomials*  $P_n^m(x)$  are solutions of the generalized Legendre's equation:

$$\frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + \left[ n(n+1) - \frac{m^2}{1-x^2} \right] y = 0$$

These polynomials can be generated by means of the formula:

$$P_n^m(x) = \frac{(-1)^m}{2^n n!} (1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n$$

Construct a user's function that allows to generate the *Associated Legendre's Polynomials* for any value of  $n$  and  $m$ , using for it the previous formula.

9. One of the functions most used in physics - mathematics are the so called *Spherical Harmonics* and they are defined by means of:

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta) e^{im\phi}$$

Construct a user's function to generate the spherical harmonics.

## 6. Conclusion

In this paper, we illustrate the use of programming with ClassPad300 calculator, when teaching some topics to students who take engineering mathematics courses. The theoretical framework for this work is the experimental mathematics. Thanks to the portable device, students have not only actively participated in the creation of algorithms and programs but also built their interests in learning more mathematics as a result. Although the paper is written for the ClassPad users, we have included the Maple program code in the supplementary material section.



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## 8. Supplementary Materials

- [1] [ClassPad 300 program code](#)
- [2] [Maple program code](#)

## 9. Acknowledgements

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